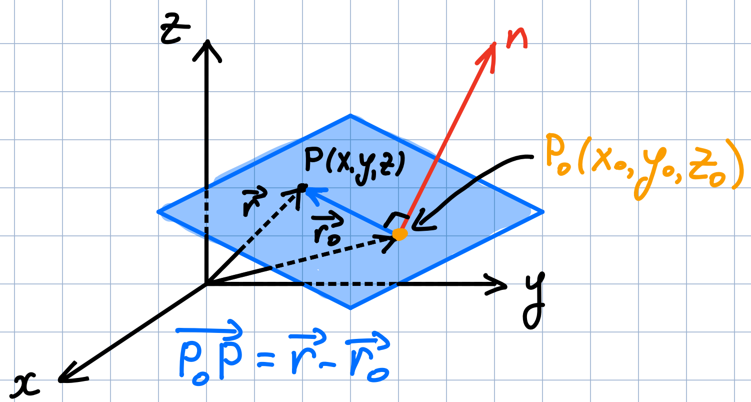
Last time: Planes



- A plane is defined by
  a point P_0 and
  an orthogonal vector $\vec{n}$
  ("the normal vector")

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0}$$ vector eq. of the plane

If $\vec{n} = \langle a, b, c \rangle$, $\vec{r} = \langle x, y, z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, then eq. of the plane is

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$ - scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$

Rmk: $\boxed{ax + by + cz + d = 0}$ - linear equation of the plane

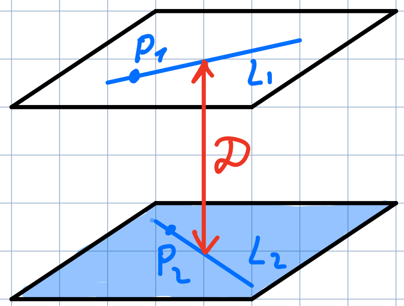
$$d = -(ax_0 + by_0 + cz_0)$$

Ex: Skew lines L_1: $x=1+t$, $y=-2+3t$, $z=4-t$

L_2: $x=2s$, $y=3+s$, $z=-3+4s$

Find distance between L_1, L_2

Solution:



1) Direction vectors: $\vec{v_1} = \langle 1, 3, -1 \rangle$

$\vec{v_2} = \langle 2, 1, 4 \rangle$

2) Common normal vector to both planes: $\vec{n} = \vec{v_1} \times \vec{v_2}$

3) Take any point on L_2 (e.g. take $s=0$) lets call it P_2

4) Plane through P_2 with a normal $\vec{n}$

5) Take any point on L_1 (e.g. take $t=0$) let's call it P_1. Find distance between P_1 and .

Lecture notes:

Marianna Russkikh

Lecture 1	Lecture 2	Lecture 3	Review 1
Lecture 4	Lecture 5	Lecture 6	

Dot & cross products review

CANVAS:

FA25-MATH-20550-SS-02

Fall Semester 2025

- Home
- Syllabus
- Announcements
- Assignments
- Modules
- Grades
- BNC First Day Course Materials

FA25-MATH-20550-SS-02 Calculus III

Day By Day Schedule | Syllabus | Class Times | Contacts and Office Hours | Homework

Video Recordings | Tutorial Woksheets | Help

Fall Semester 2025

- Home
- Syllabus
- Announcements
- Assignments
- Modules
- Grades
- BNC First Day Course Materials
- Discussions
- People
- Files
- Pages
- Outcomes
- Rubrics
- Collaborations
- Quizzes
- Settings

Chapter 12: Vetcors and Geometry of Space

- Introduction
- Lecture 1: Introduction to 3D coordinates & Vectors
- Lecture 2: Dot and Cross Product
- Lecture 3: Lines
- Lecture 4: Planes

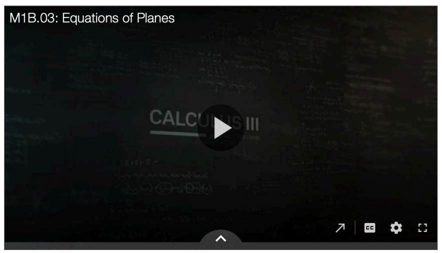
Lecture 4 introduces planes and how they interact with lines.

Companion notes: Lecture 4 Blank.pdf.

Companion notes with the solutions to the examples: Lecture 4.pdf

Video 1 | Video 2 | Video 3

Equation of a Plane



Today: Vector valued functions and space curves

The paths of objects moving through space - like the planes pictured here - can be described by vector functions.

Vector valued functions

$$\boxed{\vec{r}(t) = \langle f(t), g(t), h(t) \rangle}$$

Vector (-valued) function: domain $\longrightarrow$ range

domain: set of real numbers

range: set of vectors

$$t \longmapsto \vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$
$$= \langle f(t), g(t), h(t) \rangle$$

Rmk: domain = values t for which $\vec{r}(t)$ is defined.

Ex: $\vec{r}(t) = \langle 1+t, \frac{\sqrt{t}}{1-t}, \ln t \rangle$

$1+t$: defined for $t \in (-\infty, \infty)$

$\frac{\sqrt{t}}{1-t}$: defined for $t \in [0,1) \cup (1,\infty)$

$\ln t$: defined for $t \in (0,\infty)$

$\Rightarrow$ domain is $t \in (0,1) \cup (1,\infty)$

$f(t) = 1+t$

$g(t) = \frac{\sqrt{t}}{1-t}$

$h(t) = \ln t$

Limits of vector functions

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

provided the limits of the component functions exist

Ex: $\lim_{t\to 0} \left(\frac{\sin t}{t} \vec{i} + e^t \vec{j} + \frac{t^2+t}{t} \vec{k} \right) = 1\vec{i} + 1\vec{j} + 1\vec{k} = \langle 1, 1, 1 \rangle$

- $\vec{r}(t)$ is continuous at $\underline{a}$ if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

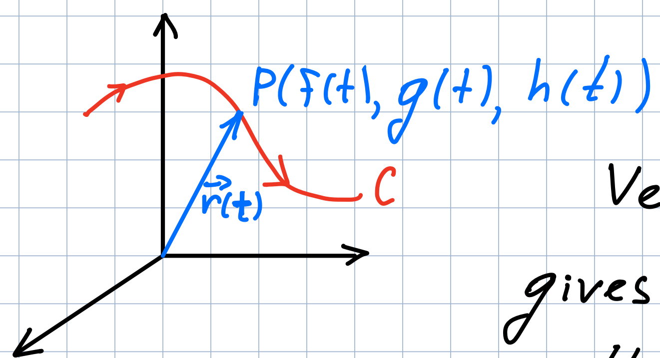
Rmk: $\vec{r}(t)$ is continuous at $\underline{a}$ $\Leftrightarrow$ f, g, h are all continuous at $\underline{a}$.

Space curves

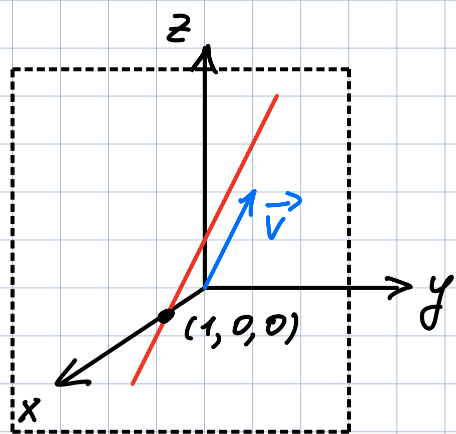
given functions

Set of points (x,y,z) with $\underbrace{x=f(t),\ y=g(t),\ z=h(t),}_{\text{parametric equations of } C,\ t\text{-parameter}}$

$t \in \underset{\text{interval}}{I}$ – "space curve" C.



Vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ gives a "moving" vector, whose tip traces the curve C.



Ex: $\vec{r}(t) = \langle 1, t, 2t \rangle$

parametric equations:

$$x = 1$$

$$y = t$$

$$z = 2t$$

line through $(1,0,0)$ parallel to $\underbrace{\langle 0, 1, 2 \rangle}_{\vec{v}}$.

Also

$\vec{r} = \vec{r}_0 + t\vec{v}$

– vector eq. of the line

$\vec{r}_0 = \langle 1, 0, 0 \rangle$

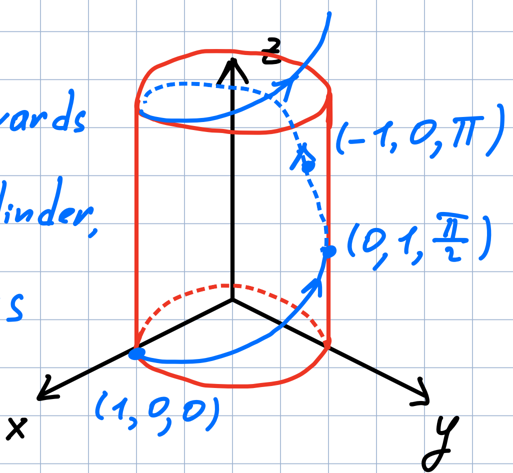
Ex: $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$ – sketch the curve

$\uparrow$ vector eq. of the curve

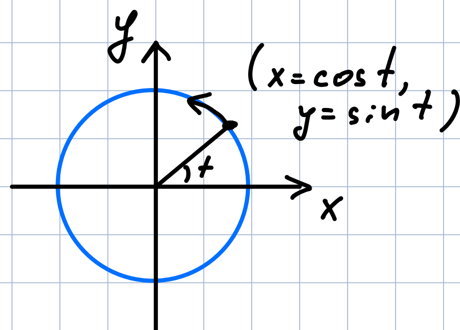
Sol: parametric eq.: $x = \cos t$, $y = \sin t$, $z = t$

1) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \implies \vec{r}(t)$ is on the cylinder $x^2 + y^2 = 1$

spirals upwards along the cylinder, as t increases



2) In xy-plane:



counterclockwise rotation

Ex: $P(1, 1, 1)$, $Q(1, 2, 3)$. Describe the line segment PQ by a vector eq.

Sol: $\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = \langle 1,\ 1+t,\ 1+2t \rangle \qquad 0 \le t \le 1$

$$\vec{r}_0 = \langle 1, 1, 1 \rangle$$

$$\vec{r}_1 = \langle 1, 2, 3 \rangle$$

Ex: Find a vec. eq. for the intersection C of the cylinder $x^2+y^2=1$ and plane $x+z=3$

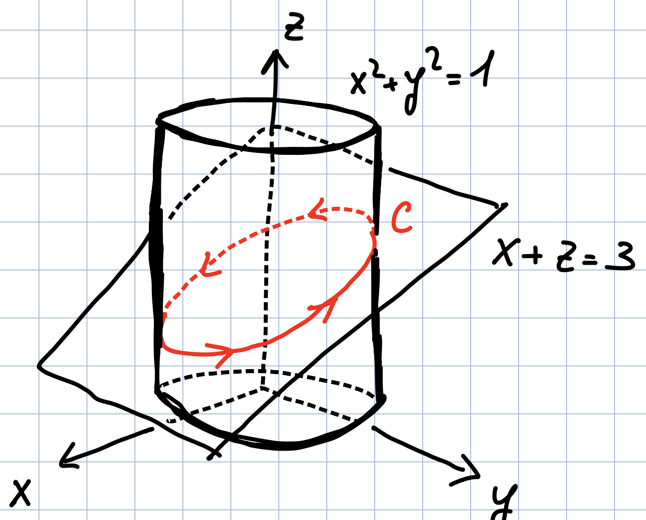
Sol: 1) projection of C onto xy-plane is the circle $\begin{cases} x^2+y^2=1 \\ z=0 \end{cases}$

given parametrically by $\begin{array}{l} x=\cos t \\ y=\sin t \\ z=0 \end{array}$, $0 \le t \le 2\pi$

2) From the eq. of the plane $z = 3 - x = 3 - \cos t$

$\Rightarrow \vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + (3-\cos t)\,\vec{k} = \langle \cos t, \sin t, 3-\cos t \rangle,$

$0 \le t \le 2\pi$



Ex: Find the intersection of C with the sphere $x^2+y^2+z^2=10$

Sol: $\cos^2 t + \sin^2 t + (3-\cos t)^2 = 10$

$1 + 9 - 6\cos t + \cos^2 t = 10$

$\cos t(\cos t - 6) = 0$

$\cos t = 0 \quad \rightarrow \quad t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Ex: Describe by vec. eq. intersection C of $z = x^2 + y^2$ - paraboloid

and $y = x^2$ - parabolic cylinder

Sol: Set $t = x$ then $y = t^2$ and $z = x^2 + y^2 = t^2 + t^4$

$$\Rightarrow \vec{r}(t) = \langle t, t^2, t^2 + t^4 \rangle$$